INJECTIVITY OF MINIMAL IMMERSIONS AND HOMEOMORPHIC EXTENSIONS TO SPACE

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ABSTRACT. We study a recent general criterion for the injectivity of the conformal immersion of a Riemannian manifold into higher dimensional Euclidean space, and show how it gives rise to important conditions for Weierstrass-Ennerper lifts defined in the unit disk $\mathbb{D}$ endowed with a conformal metric. Among the corollaries, we obtain a Becker type condition and a sharp condition depending on the Gaussian curvature and the diameter for an immersed geodesically convex minimal disk in $\mathbb{R}^3$ to be embedded. Extremal configurations for the criteria are also determined, and can only occur on a catenoid. For non-extremal configurations, we establish fibrations of space by circles in domain and range that give a geometric analogue of the Ahlfors-Weill extension.

1. INTRODUCTION

In recent years, several criteria have been derived for the injectivity of conformal immersions of planar domains into higher dimensional Euclidean spaces. Important particular cases consider Weierstrass-Enneper lifts of harmonic mappings and holomorphic immersions into $\mathbb{C}^n$ [10], [11]. The criteria represent extensions of the classical Nehari theory for homomorphic mappings in one complex variable [17], and are made possible through appropriate generalizations of the notion of a Schwarzian derivative. It is interesting to observe that the new criteria do not depend alone on the size of the generalized (conformal) Schwarzian, because the second fundamental form of the immersed surface must also be taken into account. A key ingredient in this development has been Ahlfors' definition of a Schwarzian derivative for parametrized curves in Euclidean spaces, in particular, in connection with the injectivity criterion found in [6]. This one-dimensional operator brings in both the conformal Schwarzian as well as the the second fundamental form. In [19], the author introduces the more general *Ahlfors derivative* for conformal immersions, which combines Ahlfors' Schwarzian for curves and the conformal Schwarzian. Corollary 12, on which we will concentrate, represents one of the most general formulations of a criterion for the inyectivity of a conformal immersion of

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a Riemannian manifold into Euclidean space, and we refer the reader to the paper for other interesting issues.

Our interest is the study of Corollary 12 when the Riemannian manifold takes the form of the unit disk $\mathbb{D}$ endowed with a conformal metric, and when the immersion is a Weierstrass-Enneper lift. Suitable choices of conformal metrics render, among other, generalizations of conditions by Ahlfors [3], by Becker [4], and by Epstein [16]. Moreover, it gives way to a sharp condition depending just on the Gaussian curvature and the diameter for an immersed geodesically convex minimal disk to be embedded.

As in the classical case, two additional elements appear of interest after injectivity has been established, namely, boundary behavior of the lift and possible homeomorphic or quasiconformal extensions to space. These issues have been addressed before by considering a real-valued function associated in a canonical way to the lift that measures up the conformal factor of the immersion with that of the metric [12]. We will offer a proof of Corollary 12 in the context described by appealing entirely to Ahlfors' Schwarzian for curves, showing, in passing, a crucial convexity property of the canonical function. A continuous extension to the closed disk together with the analysis when injectivity can be lost at the boundary will follow. We will apply ideas developed in [12] to define a homeomorphic extension of the lift to the entire space, as a spatial analogue of the Ahlfors-Weill construction.

The paper is organized as follows. In the remainder of the Introduction we give a brief account of the main facts about harmonic mappings and Weierstrass-Enneper lifts. In Section 2, we lay out the background material on the conformal Schwarzian that applies both for the lift and for a conformal metric in $\mathbb{D}$, making the connection with Ahlfors' derivative. Section 3 makes a summary of Ahlfors' Schwarzian for curves and the injectivity criterion derived in [6]. In Section 4 we state and prove our main result, and draw various corollaries. The analysis based on Sturm comparison and the required regularity properties of the geodesics near $\partial\mathbb{D}$ are presented in Section 5. Extremal lifts for the conditions are studied in Section 6 and the criterion involving geodesically convex minimal disk is established here. In the final section we describe the procedure that yields the homeomorphic extension to 3-space.

A planar harmonic mapping is a complex-valued harmonic function $f(z)$, $z = x+iy$, defined on some domain $\Omega \subset \mathbb{C}$. If Ω is simply connected, the mapping has a canonical decomposition $f = h+\overline{g}$, where h and g are analytic in Ω and $g(z_0) = 0$ for some specified point $z_0 \in \Omega$. The mapping f is locally univalent if and only if its Jacobian $|h'|^2 - |g'|^2$ does not vanish. It is said to be orientation-preserving if $|h'(z)| > |g'(z)|$ in Ω, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in Ω.

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h+\overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$ lifts locally to map into a minimal surface, Σ, described by conformal parameters if and only if its dilatation $\omega = q^2$, the square of a meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \mathrm{Re}\{f(z)\}\,, \quad V(z) = \mathrm{Im}\{f(z)\}\,, \quad W(z) = 2\,\mathrm{Im}\left\{\int_{z_0}^{z} h'(\zeta)q(\zeta)\,d\zeta\right\}.$$

We use the notation

$$\widetilde{f}(z) = \big(U(z), V(z), W(z)\big)$$

for the lifted mapping of Ω into Σ. The height of the surface can be expressed more symmetrically as

$$W(z) = 2\,\mathrm{Im}\left\{\int_{z_0}^{z} \sqrt{h'(\zeta)g'(\zeta)}\,d\zeta\right\},$$

since a requirement equivalent to $\omega = q^2$ is that $h'g'$ be the square of an analytic function. The first fundamental form of the surface is $ds^2 = e^{2\sigma}|dz|^2$, where the conformal factor is

$$e^{\sigma} = |h'| + |g'|\,.$$

The Gauss curvature of the surface at a point $\widetilde{f}(z)$ for which $h'(z) \neq 0$ is

$$K = -e^{-2\sigma}\Delta\sigma = -\frac{4|q'|^2}{|h'|^2(1+|q|^2)^4}\,, \tag{1.1}$$

where Δ is the Laplacian operator. Further information about harmonic mappings and their relation to minimal surfaces can be found in [15].

For a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$, whose dilatation is the square of a meromorphic function, we have defined [8] the *Schwarzian derivative* by the formula

$$\mathcal{S}f = 2\big(\sigma_{zz} - \sigma_z^2\big)\,, \tag{1.2}$$

where

$$\sigma_z = \frac{\partial\sigma}{\partial z} = \frac{1}{2}\left(\frac{\partial\sigma}{\partial x} - i\frac{\partial\sigma}{\partial y}\right), \qquad z = x + iy\,.$$

Some background for this definition is discussed in Section 2. With $h'(z) \neq 0$ and $g'/h' = q^2$, a calculation (*cf.* [8]) produces the expression

$$\mathcal{S}f = \mathcal{S}h + \frac{2\overline{q}}{1+|q|^2}\left(q'' - \frac{q'h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2.$$

As observed in [8], the formula remains valid if ω is not a perfect square, provided that neither h' nor g' has a simple zero.

It must be emphasized that we are not requiring our harmonic mappings to be locally univalent. In other words, the Jacobian need not be of constant sign in the domain Ω. The orientation of the mapping may reverse, corresponding to a folding in the associated minimal surface. It is also possible for the minimal surface to exhibit several sheets above a point in the (U,V)–plane. Thus the lifted mapping $\widetilde{f}$ may be univalent even when the underlying mapping f is not.

2. Conformal Schwarzian and Ahlfors' Derivative

In this section, we present the definition of the conformal Schwarzian that applies to immersions and to conformal metrics. We will also present its relation to the Ahlfors derivative introduced in [19], when dealing with Weierstrass-Enneper lifts of harmonic mappings. The use of Ahlfors' Schwarzian for curves in the proof of Theorem 4.1 below avoids the need to consider a Schwarzian derivative of harmonic mappings *relative* to conformal metrics in $\mathbb{D}$. Nevertheless, the first term on the left-hand side in (4.1) below, indeed corresponds to the Schwarzian of f relative to the conformal metric in $\mathbb{D}$, making the connections with Corollary 12 in [19].

Furthermore, despite the apparent Euclidean nature of Ahlfors' Schwarzian, the chain rule and the natural parametrizations of geodesics in the conformal geometry lead to the required lower bounds for the Hessian of the canonical function *relative to the conformal metric* that are of central use in Section 5 and 7.

The definition of conformal Schwarzian and its properties are suggested by the classical case, and have analogues there, but the generalization must be framed in the terminology of differential geometry. We refer to [18] for the higher dimensional setting and to [7] for applications of convexity in 2 dimensions, similar to what we will do here for harmonic mappings.

Let $\mathbf{g}$ be a Riemannian metric on the disk $\mathbb{D}$. We may assume that $\mathbf{g}$ is conformal to the Euclidean metric, $\mathbf{g}_0 = dx\otimes dx + dy\otimes dy = |dz|^2$. Let ψ be a smooth function on $\mathbb{D}$ and form the symmetric 2-tensor

$$\mathrm{Hess}_{\mathbf{g}}(\psi) - d\psi\otimes d\psi. \tag{2.1}$$

Here Hess denotes the Hessian operator. For example, if $\gamma(s)$ is an arc-length parametrized geodesic for $\mathbf{g}$, then

$$\mathrm{Hess}_{\mathbf{g}}(\psi)(\gamma',\gamma') = \frac{d^2}{ds^2}(\psi\circ\gamma)\,.$$

The Hessian depends on the metric, and since we will be changing metrics we indicate this dependence by the subscript $\mathbf{g}$.

With some imagination the tensor (2.1) begins to resemble a Schwarzian; among other occurrences in differential geometry, it arises (in 2 dimensions) if one differentiates the equation that relates the geodesic curvatures of a curve for two conformal metrics. Such a curvature formula is a classical interpretation of the Schwarzian derivative, see [18] and [9]. The trace of the tensor is the function

$$\frac{1}{2}(\Delta_{\mathbf{g}}\psi - ||\,\mathrm{grad}_{\mathbf{g}}\,\psi||^2_{\mathbf{g}}),$$

where again we have indicated by a subscript that the Laplacian, gradient and norm all depend on $\mathbf{g}$. It turns out to be most convenient to work with a traceless tensor when generalizing the Schwarzian, so we subtract off this function times the metric $\mathbf{g}$ and define the *Schwarzian tensor* to be the symmetric, traceless, 2-tensor

$$B_{\mathbf{g}}(\psi) = \mathrm{Hess}_{\mathbf{g}}(\psi) - d\psi \otimes d\psi - \frac{1}{2}(\Delta_{\mathbf{g}}\psi - ||\,\mathrm{grad}_{\mathbf{g}}\,\psi||^2)\mathbf{g}\,.$$

Working in standard Cartesian coordinates one can represent $B_{\mathbf{g}}(\psi)$ as a symmetric, traceless 2×2 matrix, say of the form

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}.$$

Further identifying such a matrix with the complex number $a + bi$ then allows us to associate the tensor $B_{\mathbf{g}}(\psi)$ with $a + bi$.

At each point $z \in \mathbb{D}$, the expression $B_{\mathbf{g}}(\psi)(z)$ is a bilinear form on the tangent space at z, and so its norm is

$$||B_{\mathbf{g}}(\psi)(z)||_{\mathbf{g}} = \sup_{X,Y} B_{\mathbf{g}}(\psi)(z)(X, Y)\,,$$

where the supremum is over unit vectors in the metric $\mathbf{g}$. If we compute the tensor with respect to the Euclidean metric and make the identification with a complex number as above, then

$$||B_{\mathbf{g_0}}(\psi)(z)||_{\mathbf{g_0}} = |a + bi|\,.$$

Now, if f is analytic and locally univalent in $\mathbb{D}$, then it is a conformal mapping of $\mathbb{D}$ with the metric $\mathbf{g}$ into $\mathbb{C}$ with the Euclidean metric. The pullback $f^*\mathbf{g}_0$ is a metric on $\mathbb{D}$ conformal to $\mathbf{g}$, say $f^*\mathbf{g}_0 = e^{2\psi}\mathbf{g}$, and the (conformal) Schwarzian of f is now defined to be

$$\mathcal{S}_{\mathbf{g}}f = B_{\mathbf{g}}(\psi)\,.$$

If we take $\mathbf{g}$ to be the Euclidean metric then $\psi = \log|f'|$. Computing $B_{\mathbf{g}_0}(\log|f'|)$ and writing it in matrix form as above results in

$$B_{\mathbf{g}_0}(\log|f'|) = \begin{pmatrix} \mathrm{Re}\,\mathcal{S}f & -\mathrm{Im}\,\mathcal{S}f \\ -\mathrm{Im}\,\mathcal{S}f & -\mathrm{Re}\,\mathcal{S}f \end{pmatrix},$$

where $\mathcal{S}f$ is the classical Schwarzian derivative of f. In this way we identify $B_{\mathbf{g}_0}(\log|f'|)$ with $\mathcal{S}f$.

Next, if $f = h + \overline{g}$ is a harmonic mapping of $\mathbb{D}$ and $\sigma = \log(|h'| + |g'|)$ is the conformal factor associated with the lift $\widetilde{f}$, we put

$$\mathcal{S}f = \mathcal{S}_{\mathbf{g}_0}\widetilde{f} = B_{\mathbf{g}_0}(\sigma).$$

Calculating this out and making the identification of the generalized Schwarzian with a complex number produces

$$B_{\mathbf{g}_0}(\sigma) = 2(\sigma_{zz} - \sigma_z^2)\,,$$

which is the definition of $\mathcal{S}f$ given in (1.2).

In this context, the Ahlfors derivative $\mathcal{A}f$ relative to $\mathbf{g}_0$ introduced in [19] is related to the conformal Schwarzian by the equation

$$\mathcal{A}f = \mathcal{S}f + \frac{1}{2}|K \circ \widetilde{f}|\mathbf{g}_0 \,.$$

The definition of $\mathcal{A}f$ gives a two-tensor for arbitrary conformal immersions, following partly the conformal Schwarzian, but it incorporates information of the second fundamental form of the target when codimension exists. It gives back Ahlfors Schwarzian for curves (presented in the next section) when the domain manifold is an interval, and vanishes for Möbius transformations of $\mathbb{R}^n$. As pointed out by the author in [19], it is interesting that no such operator will exhibit in addition a general chain rule $\mathcal{A}(G \circ F) = \mathcal{A}F + F^*(\mathcal{A}G)$, although the operator introduced will comply with this chain rule in many situations. We refer the reader to [19] for the analysis leading to the definition and for further details.

3. Ahlfors' Schwarzian

Ahlfors [1] introduced a notion of Schwarzian derivative for mappings of a real interval into $\mathbb{R}^n$ by formulating suitable analogues of the real and imaginary parts of $\mathcal{S}f$ for analytic functions f. A simple calculation shows that

$$\mathrm{Re}\{\mathcal{S}f\} = \frac{\mathrm{Re}\{f'''\overline{f'}\}}{|f'|^2} - 3\,\frac{\mathrm{Re}\{f''\overline{f'}\}^2}{|f'|^4} + \frac{3}{2}\frac{|f''|^2}{|f'|^2}\,.$$

For mappings $\varphi : (a,b) \to \mathbb{R}^n$ of class C^3 with $\varphi'(x) \neq 0$, Ahlfors defined the analogous expression

$$\mathcal{S}_1\varphi = \frac{\varphi''' \cdot \varphi'}{|\varphi'|^2} - 3\frac{(\varphi'' \cdot \varphi')^2}{|\varphi'|^4} + \frac{3}{2}\frac{|\varphi''|^2}{|\varphi'|^2}\,, \tag{3.1}$$

where $\cdot$ denotes the Euclidean inner product and now $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$. Ahlfors also defined a second expression analogous to $\mathrm{Im}\{\mathcal{S}f\}$, but this is not relevant to the present discussion.

Ahlfors' Schwarzian is invariant under postcomposition with Möbius transformations; that is, under every composition of rotations, magnifications, translations, and inversions in $\mathbb{R}^n$. Only its invariance under inversion

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2}\,, \qquad \mathbf{x} \in \mathbb{R}^n\,,$$

presents a difficulty; this can be checked by straightforward but tedious calculation. It should also be noted that $\mathcal{S}_1$ transforms as expected under change of parameters. If $x = x(t)$ is a smooth function with $x'(t) \neq 0$, and $\psi(t) = \varphi(x(t))$, then

$$\mathcal{S}_1\psi(t) = \mathcal{S}_1\varphi(x(t))\,x'(t)^2 + \mathcal{S}x(t)\,. \tag{3.2}$$

With the notation $v = |\varphi'|$, and based on the Frenet–Serret formulas, it was shown in [6] that

$$\mathcal{S}_1\varphi = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2 = \mathcal{S}(s) + \frac{1}{2}v^2k^2\,, \tag{3.3}$$

where $s = s(x)$ is the arc-length of the curve and k is its curvature. Our proof of Theorem 1 will be based on the following result found in [6].

Theorem A. Let $p(x)$ be a continuous function such that the differential equation $u''(x)+p(x)u(x) = 0$ admits no nontrivial solution $u(x)$ with more than one zero in $(-1, 1)$. Let $\varphi : (-1, 1) \to \mathbb{R}^n$ be a curve of class C^3 with tangent vector $\varphi'(x) \neq 0$. If $\mathcal{S}_1\varphi(x) \leq 2p(x)$, then φ is univalent.

If the function $p(x)$ of Theorem A is even, then the solution u_0 of the differential equation $u'' + pu = 0$ with initial conditions $u_0(0) = 1$ and $u_0'(0) = 0$ is also even, and therefore $u_0(x) \neq 0$ on $(-1, 1)$, since otherwise it would have at least two zeros. Thus the function

$$\Phi(x) = \int_0^x u_0(t)^{-2}\,dt\,, \qquad -1 < x < 1\,, \tag{3.4}$$

is well defined and has the properties $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi''(0) = 0$, $\Phi(-x) = -\Phi(x)$. The standard method of reduction of order produces the independent solution $u = u_0\Phi$ to $u''+pu = 0$, and so $\mathcal{S}\Phi = 2p$. Note also that $\mathcal{S}_1\Phi = \mathcal{S}\Phi$, since Φ is real-valued. Thus $\mathcal{S}_1\Phi = 2p$.

The next theorem, again to be found in [6], asserts that the mapping $\Phi : (-1, 1) \to \mathbb{R} \subset \mathbb{R}^n$ is extremal for Theorem A if $\Phi(1) = \infty$, and that every extremal mapping φ is then a Möbius postcomposition of Φ.

Theorem B. Let $p(x)$ be an even function with the properties assumed in Theorem A, and let Φ be defined as above. Let $\varphi : (-1, 1) \to \mathbb{R}^n$ satisfy $\mathcal{S}_1\varphi(x) \leq 2p(x)$ and have the normalization $\varphi(0) = 0$, $|\varphi'(0)| = 1$, and $\varphi''(0) = 0$. Then $|\varphi'(x)| \leq \Phi'(|x|)$ for $x \in (-1, 1)$, and φ has an extension to the closed interval $[-1, 1]$ that is continuous with respect to the spherical metric. Furthermore, there are two possibilities, as follows.

(i) If $\Phi(1) < \infty$, then φ is univalent in $[-1, 1]$ and $\varphi([-1, 1])$ has finite length.

(ii) If $\Phi(1) = \infty$, then either φ is univalent in $[-1, 1]$ or $\varphi = R \circ \Phi$ for some rotation R of $\mathbb{R}^n$.

Note that in case (ii) the mapping Φ sends both ends of the interval to the point at infinity and is therefore not univalent in $[-1, 1]$. The role of Φ as an extremal for the harmonic univalence criterion (4.1) will emerge in the following sections. Two important corollaries of Theorem B are that a curve $\varphi : (-1, 1) \to \mathbb{R}^n$ satisfying $\mathcal{S}_1\varphi(x) \leq 2p(x)$ for which $\varphi(1) = \varphi(-1)$ must take the closed interval $[-1, 1]$ to a circle or a line union the point at infinity, and that $\mathcal{S}_1\varphi \equiv 2p$.

Remark 3.1. A final important observation that will be used in Section 6, is that if $\mathcal{S}_1\varphi < 2\pi^2/l^2$ on an interval I of length l, then φ is injective on the closed interval $\overline{I}$. To prove this, we may assume that $I = (-l/2, l/2)$. Note that $p(x) \equiv \pi^2/l^2$ satisfies the hypothesis in Theorem A. The even solution u_0 above is given by $\cos(cx)$, with $c = \pi/l$, and corresponding extremal $\Phi(x) = (1/c)\tan(cx)$ for which $\mathcal{S}\Phi = 2\pi^2/l^2$.

The estimate $\mathcal{S}_1\varphi < 2\pi^2/l^2$ and Theorem A show that φ is injective on I, and Theorem B shows that the extension to $\overline{I}$ must remain injective, for otherwise $\mathcal{S}_1\varphi = 2\pi^2/l^2$, a contradiction.

4. Embedded minimal disks

In this section we shall give a proof of the following criterion for the conformal parametrization of minimal disks to be injective. This theorem corresponds to Corollary 12 in [19] when the domain manifold $M = \mathbb{D}$ endowed with the conformal metric $\mathbf{g}$. Follow up results having to do with continuous extension to the closed disk, extremal configuration, and homeomorphic extension to all 3-space, will be studied in the final sections.

Theorem 4.1. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Let $\mathbf{g} = e^{2\rho}\mathbf{g}_0$ be a metric in $\mathbb{D}$ conformal to the Euclidean metric, and suppose that any two points in $\mathbb{D}$ can be joined by a geodesic in the metric $\mathbf{g}$ of length less that δ, for some $0 < \delta \leq \infty$. If*

$$\left|\mathcal{S}f - 2\left(\rho_{zz} - \rho_z^2\right)\right| + e^{2\sigma}|K| \leq \frac{2\pi^2 e^{2\rho}}{\delta^2} + 2\rho_{z\bar{z}} \tag{4.1}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

The proof will be based on showing that under (4.1), the restriction of $\widetilde{f}$ to any geodesic is injective. To this end, we state without proof the following variant of Lemma 2 in [11].

Lemma 4.2. *Let $\widetilde{f} : \mathbb{D} \to \Sigma$ be the lift of a harmonic mapping f defined in $\mathbb{D}$. Let $\gamma(t)$ be a Euclidean arc-length parametrized curve in $\mathbb{D}$ with curvature $\kappa(t)$, and let $\varphi(t) = \widetilde{f}(\gamma(t))$ be the corresponding parametrization of $\Gamma = \widetilde{f}(\gamma)$ on Σ. Let $V(t)$ be the Euclidean unit tangent vector field along $\varphi(t)$, given by*

$$V(t) = \frac{\varphi'(t)}{|\varphi'(t)|}\,.$$

If II stands for the second fundamental form on Σ then

$$\mathcal{S}_1\varphi = \mathrm{Re}\{\mathcal{S}f(\gamma)(\gamma')^2\} + \frac{1}{2}e^{2\sigma(\gamma)}\left(|K(\varphi)| + |II(V,V)|^2\right) + \frac{1}{2}\kappa^2\,. \tag{4.2}$$

With this, we can now prove Theorem 4.1.

Proof of Theorem 4.1. Let $z_1, z_2 \in \mathbb{D}$ be two points. By assumption, there is a geodesic γ in the conformal metric $e^{2\rho}\mathbf{g}_0$ of length $< \delta$ joining the two points. Let $\gamma = \gamma(t)$ be a Euclidean arc-length parametrization, and let $t = t(s)$ be a change of parameters so that $s \to \gamma(t(s))$ is a unit-length parametrization relative of the background metric. This means that

$$e^{\rho(\gamma(t(s)))}|\gamma'(t(s))|\frac{dt}{ds} = 1\,,$$

or

$$\frac{dt}{ds} = e^{-\rho(\gamma(t(s)))}\,.$$

We shall compute Ahlfors' Schwarzian of the parametrization $\psi(s) = \widetilde{f}(\gamma(t(s))) = \varphi(t(s))$, using the notation of Lemma 4.2. The new parametrization is defined for s on an interval I of length less than δ. Using (3.2), we have that

$$\mathcal{S}_1\psi(s) = \mathcal{S}_1\varphi(t(s))(t'(s))^2 + \mathcal{S}t(s)\,,$$

where the first term on the right hand side comes from Lemma 4.2. We compute now the second term:

$$\mathcal{S}t(s) = \left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2\,.$$

We introduce the Euclidean unit vectors $\hat{t}, \hat{n}$ given by $\hat{t} = \gamma'(t)$ and $\gamma''(t) = \kappa\hat{n}$. Since $t' = e^{-\rho(\gamma)}$, and since $(dt/ds) = e^{-\rho}$, we see that

$$\frac{t''}{t'} = -e^{-\rho}\nabla\rho\cdot\hat{t}\,,$$

and therefore

$$\left(\frac{t''}{t'}\right)' = -e^{-2\rho}\,\mathrm{Hess}(\rho)(\hat{t},\hat{t}) - e^{-2\rho}(\nabla\rho\cdot\hat{n})\kappa + e^{-2\rho}(\nabla\rho\cdot\hat{t})^2\,.$$

Because γ is a geodesic in the conformal metric, we have that $\kappa = \nabla\rho\cdot\hat{n}$, which gives

$$e^{2\rho}\mathcal{S}t(s) = -\,\mathrm{Hess}(\rho)(\hat{t},\hat{t}) + \frac{1}{2}(\nabla\rho\cdot\hat{t})^2 - \kappa^2\,. \tag{4.3}$$

Therefore

$$e^{2\rho}\mathcal{S}_1\psi = \mathrm{Re}\{\mathcal{S}f(\gamma)(\gamma')^2\} + \frac{1}{2}e^{2\sigma}\left(|K| + |II(V,V)|^2\right) + A\,,$$

where

$$A = -\,\mathrm{Hess}(\rho)(\hat{t},\hat{t}) + \frac{1}{2}(\nabla\rho\cdot\hat{t})^2 - \frac{1}{2}(\nabla\rho\cdot\hat{n})^2\,.$$

Using that

$$B_{\mathbf{g_0}}(\rho)(\hat{t},\hat{t}) = \mathrm{Hess}(\rho)(\hat{t},\hat{t}) - (\nabla\rho\cdot\hat{t})^2 - \frac{1}{2}\left(\Delta\rho - |\nabla\rho|^2\right)\,,$$

some algebraic manipulations give that

$$A = -B_{\mathbf{g_0}}(\rho)(\hat{t},\hat{t}) - 2\rho_{z\bar{z}} = -\mathrm{Re}\{2(\rho_{zz}-\rho_z^2)(\gamma')^2\} - 2\rho_{z\bar{z}}\,.$$

Therefore

$$\begin{aligned} e^{2\rho}\mathcal{S}_1\psi &= \mathrm{Re}\left\{\left(\mathcal{S}f(\gamma) - 2(\rho_{zz}-\rho_z^2)\right)(\gamma')^2\right\} + \frac{1}{2}e^{2\sigma}\left(|K| + |I\!I(V,V)|^2\right) - 2\rho_{z\bar{z}} \\ &\le \left|\mathcal{S}f(\gamma) - 2(\rho_{zz}-\rho_z^2)\right| + e^{2\sigma}|K| - 2\rho_{z\bar{z}}\,. \end{aligned}$$

Finally, the inequality (4.1) implies that

$$\mathcal{S}_1\psi \le \frac{2\pi^2}{\delta^2}\,. \tag{4.4}$$

We appeal now to Theorem A with $p(x) = \pi^2/\delta^2$ on an open interval J of length less than δ containing the closed interval $\overline{I}$, to conclude that ψ is injective. This proves the theorem.

□

We will draw some corollaries as particular important cases of this theorem. The first instance corresponds to the case when the diameter $\delta = \infty$, that is, when the metric is complete.

Corollary 4.3. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Let $\mathbf{g} = e^{2\rho}\mathbf{g}_0$ be a complete metric in $\mathbb{D}$ conformal to the Euclidean metric. If*

$$\left|\mathcal{S}f - 2\left(\rho_{zz}-\rho_z^2\right)\right| + e^{2\sigma}|K| \le -\frac{1}{2}\rho_{z\bar{z}} \tag{4.5}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

A second set of cases arise when considering

$$e^{\sigma} = \frac{1}{(1-|z|^2)^t} \quad , \ t \ge 0\,.$$

The resulting conformal metrics have negative curvature and are complete for $t \ge 1$. For $0 \le t < 1$ the disk is still geodesically convex but has finite diameter given by

$$\delta = 2\int_0^1 \frac{dx}{(1-x^2)^t}\,,$$

which can be expressed in terms of the Gamma function as

$$\delta = \sqrt{\pi}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2}-t)}\,.$$

Corollary 4.4. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. If either*

$$\left|\mathcal{S}f - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2}\right| + e^{2\sigma}|K| \leq \frac{2t}{(1-|z|^2)^2} \ \ , \ t \geq 1 \tag{4.6}$$

or

$$\left|\mathcal{S}f - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2}\right| + e^{2\sigma}|K| \leq \frac{2t}{(1-|z|^2)^2} + \frac{2\pi}{(1-|z|^2)^{2t}}\left(\frac{\Gamma(\frac{3}{2}-t)}{\Gamma(1-t)}\right)^2 \ \ , \ 0 \leq t < 1 \tag{4.7}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

Three important instances of this corollary are obtained when setting $t = 0, t = 1$ and $t = 2$, which yields, respectively,

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq \frac{\pi^2}{2}\,, \tag{4.8}$$

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq \frac{2}{(1-|z|^2)^2}\,, \tag{4.9}$$

and

$$\left|\mathcal{S}f + \frac{4\bar{z}^2}{(1-|z|^2)^2}\right| + e^{2\sigma}|K| \leq \frac{4}{(1-|z|^2)^2}$$

as sufficient conditions for injectivity.

The third condition implies that

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq \frac{4}{1-|z|^2} \tag{4.10}$$

is sufficient for the injectivity of the lift. Indeed, if (4.10) holds, then

$$\begin{aligned}\left|\mathcal{S}f + \frac{4\bar{z}^2}{(1-|z|^2)^2}\right| + e^{2\sigma}|K| &\leq |\mathcal{S}f| + \frac{4|z|^2}{(1-|z|^2)^2} + e^{2\sigma}|K| \\ &\leq \frac{4}{1-|z|^2} + \frac{4|z|^2)}{(1-|z|^2)^2} = \frac{4}{(1-|z|^2)^2}\,.\end{aligned}$$

Conditions (4.8), (4.9), and (4.10) were obtained in [10].

The criteria (4.10) for $1 \leq t \leq 2$ and (4.11) without the diameter term are generalizations of Ahlfors' condition for holomorphic f

$$\left|\mathcal{S}f - \frac{2c(1-c)\bar{z}^2}{(1-|z|^2)^2}\right| \leq \frac{2|c|}{(1-|z|^2)^2} \tag{4.11}$$

when c is real. In (4.14) c may be any complex number with $|c-1| < 1$ [2].

We draw here two additional corollaries from our main result.

Corollary 4.5. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Let $\tau = \tau(z)$ be a real-valued function in $\mathbb{D}$ satisfying*

$$|\tau_z| \leq \frac{c}{1-|z|^2} \quad , \ z \in \mathbb{D}\,, \tag{4.12}$$

for some constant $c < 1$. If

$$\left|\mathcal{S}f - 2(\tau_{zz} - \tau_z^2) + \frac{4\bar{z}\tau_z}{1-|z|^2}\right| + e^{2\sigma}|K| \leq \frac{2(1+(1-|z|^2)^2\tau_{z\bar{z}})}{(1-|z|^2)^2}\,, \tag{4.13}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

This corollary constitutes a generalization of one of the main results in [16]. See also [3] for even more general sufficient criteria for holomorphic mappings to be injective.

Proof. Let $e^\rho = e^\tau/(1-|z|^2)$. The inequality (4.15) guarantees that the radial derivative ρ_r is positive for all r sufficiently close to 1. This implies that $\mathbb{D}$ is geodesically convex in the metric $e^{2\rho}\mathbf{g}_0$, and thus Theorem 4.2 is applicable. In this corollary we have excluded the diameter term appearing in (4.1).

□

Corollary 4.6. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Suppose that*

$$|\sigma_z| \leq \frac{c}{1-|z|^2} \quad , \ z \in \mathbb{D}\,,$$

for some constant $c < 1$. If

$$|z\sigma_z| + \frac{1}{4}(1-|z|^2)e^{2\sigma}|K| \leq \frac{1}{1-|z|^2}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

Corollary 4.6 can be considered a generalization of the well known criterion for univalence of Becker [4].

Proof. The condition on σ_z ensures as before that $\mathbb{D}$ is geodesically convex with the metric $e^{2\sigma}\mathbf{g}_0$. The corollary follows at once by applying Theorem 4.2 with $e^\rho = e^\sigma/(1-|z|^2)$. □

5. Convexity and Continuous Extensions

The purpose of this section is to establish the continuous extension to $\overline{\mathbb{D}}$ of lifts satisfying (4.1). We begin with the following crucial lemma.

Lemma 5.1. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Let $\mathbf{g} = e^{2\rho}\mathbf{g}_0$ be a metric in $\mathbb{D}$ conformal to the Euclidean metric, and suppose that (4.1) holds. Then*

$$u_f(z) = \sqrt{e^{\rho-\sigma}}$$

satisfies

$$\frac{d^2}{ds^2}u_f(\gamma(s)) + \frac{\pi^2}{\delta^2}u_f(\gamma(s)) \geq 0\,,$$

for any arc-length parametrized geodesic $\gamma(s)$ in the metric $\mathbf{g}$. In particular, when $\mathbf{g}$ is complete then u_f is convex in this metric.

Proof. Let $\gamma = \gamma(s)$ be a geodesic in $\mathbf{g}$ parametrized by arc-length, as was considered in the proof of Theorem 4.2, where it was shwon that the curve $\psi(s) = \widetilde{f}(\gamma(s))$ satisfies (4.4). Let $\tau = \tau(s)$ be such that $\tau'(s) = |\psi'(s)| = e^{\sigma-\rho}$. A well known fact that is easy to verify, ensures that the positive function $U = (\tau')^{-1/2} = u_f(\gamma)$ satisfies

$$U'' + \frac{1}{2}(S\tau)U = 0\,.$$

Because of the definition of the $\mathcal{S}_1$ operator, we have that $S\tau \leq \mathcal{S}_1\psi \leq 2\pi^2/\delta^2$, and thus

$$U'' + \frac{\pi^2}{\delta^2}U \geq 0\,,$$

as desired. □

By means of comparison we will obtain from this lemma lower bounds for the canonical function u_f along geodesics, which will ensure an extension of the lift along geodesics. A bit more will be required to turn this information into a continuous extension to the closed disk $\overline{\mathbb{D}}$. To this end, we introduce three conditions on the metric $\mathbf{g}$ that control the geometry of the geodesics near $\partial\mathbb{D}$. The conditions are mild and far from restrictive, and were considered in [7] for the same purpose in the context of general criteria for injectivity for holomorphic functions defined in $\mathbb{D}$.

Unless noted otherwise, in the remainder of this paper we will *always assume* that the metric $\mathbf{g}$ has non-positive curvature, that is, that $\rho_{z\bar{z}} \geq 0$, and so we will not state this as a separate assumption in any of our results. Geometrically, the main consequence of this is that geodesics cannot cross more than once in $\mathbb{D}$. We let $l_{\mathbf{g}}$ denote the length function (of a curve) and $d_{\mathbf{g}}$ the distance (between points).

The first property has to do with extending geodesics to the boundary, and with reaching every boundary point in this way. We state the property first as it often appears in the literature, but we must then say more to distinguish the complete and the non-complete cases.

Definition 5.2. The metric $\mathbf{g}$ on $\mathbb{D}$ has the *Unique Limit Point* property (ULP) if:

(a) Let $z_0 \in \mathbb{D}$. If $\gamma(t)$, $0 \le t < T \le \infty$ is a maximally extended geodesic starting at z_0 then $\lim_{t\to T}\gamma(t)$ exists (in the Euclidean sense). We denote it by $\gamma(T) \in \partial\mathbb{D}$.

(b) The limit point is a continuous function of the initial direction at z_0.

(c) Let $\zeta \in \partial\mathbb{D}$. Then there is a geodesic starting at z_0 whose limit point on $\partial\mathbb{D}$ is ζ.

We say a little more about part (c) in this condition. The assumption of non-positive curvature implies that the limit point is a monotonic function of the initial direction at the base point. Part (b) requires that it is continuous. It is conceivable that, for some metrics, all geodesics from a base point might tend to the same limit point on the boundary, so the mapping from initial directions to points on $\partial\mathbb{D}$ would reduce to a constant. We want to avoid this degenerate situation and be certain that every boundary point is 'visible', so we include that fact in the statement of (ULP).

(ULP) is a natural condition on complete metrics and is frequently formulated this way, if not with this appellation. For our work on boundary behavior in the non-complete case we have to strengthen it slightly. Again take any base point $z_0 \in \mathbb{D}$ and consider geodesics from z_0 extended maximally to their unique limit points on the boundary. In general, the length of such a geodesic as a function of the initial direction at z_0 is lower semicontinuous, and for our arguments we need to know that it is continuous. We let (ULP*) mean (ULP) *plus* the continuity of the length function. This is the assumption we will often adopt in the non-complete case. In the complete case the length function is the constant function $+\infty$ and the particular problems we encounter in the non-complete case do not come up; (ULP) will suffice as is.

The conditions above must be hypotheses in many of our results, but none of them, alone or together, is asking too much of a metric (see, for example, Theorems 7, 8 in [7].)

Theorem 5.3. *Let f be a harmonic mapping satisfying the hypotheses in Theorem 4.1, and suppose that the metric $\mathbf{g}$ satisfies (ULP) if it is complete and (ULP*) if it is not complete. Then $\widetilde{f}$ admits a (spherically) continuous extension to $\overline{\mathbb{D}}$.*

Proof. Let $\Sigma = \widetilde{f}(\mathbb{D})$. The proof is based entirely on the one given for the corresponding theorem for holomorphic mappings in [7, Thm. 3]. We include the proof for the convenience of the reader. We will show that small arcs on S^1, corresponding to intervals of initial directions of geodesics from a base point, which parametrize small arcs on $\partial\Sigma$. To obtain the requisite estimates we have to modify $\widetilde{f}$ by Möbius transformations of the range, and this is why the theorem is stated in terms of spherical continuity. Composing $\widetilde{f}$ with a Möbius transformation will generally not preserve minimality, but as we have seen, all proof are based on Ahfors' operator, which is preserved under such compositions.

The proof is slightly different in the two cases $\delta < \infty$ and $\delta = \infty$. We consider first $\delta < \infty$; thus (ULP*) is in force. Let $\zeta_0 \in \partial\mathbb{D}$ and let γ_0 be a geodesic in $\mathbb{D}$ ending at ζ_0. Let $z_0 \in \gamma_0$ be a point of distance $< \delta/8$ from ζ_0, and let θ_0 be the direction of γ_0 at z_0. Choose a small enough neighborhood V of initial directions about θ_0 with corresponding geodesics covering an arc $I \subset \partial\mathbb{D}$ of limit points so that the distances between z_0 and all such limit points is $\leq \delta/4$.

Let $\theta \in V$ and let $\gamma(t)$, $0 \leq t \leq T_\theta$ be the corresponding geodesic starting at z_0 and ending at a point on $I \subset \partial\mathbb{D}$. Replace $\widetilde{f}$ by $M \circ \widetilde{f}$, where the Möbius transformation M is chosen so that the associatied function $u_{M\circ\widetilde{f}}$ satisfies

$$\operatorname{grad} u_{M\circ\widetilde{f}}(z_0) = 0 \quad \text{and} \quad u_{M\circ f}(z_0) = 1.$$

We want to apply Lemma 5.1 to $u_{M\circ\widetilde{f}}$ along the geodesics γ. Since $\mathcal{S}_1(M\circ\widetilde{f}) = \mathcal{S}_1\widetilde{f}$, we continue to write $\widetilde{f}$ for $M \circ \widetilde{f}$ and $u_{\widetilde{f}}$ for $u_{M\circ\widetilde{f}}$. The function $U(t) = u_{\widetilde{f}}(\gamma(t))$ satisfies

$$U'' \geq -\frac{\pi^2}{\delta^2}U, \quad U(0) = 0, \quad U'(0) = 1.$$

From this,

$$U(t) \geq \cos(\frac{\pi}{\delta}t),$$

and so

$$U(t) \geq \cos(\frac{\pi}{\delta}\frac{\delta}{4}) = \frac{1}{\sqrt{2}}.$$

Note that since $u_{\widetilde{f}}$ is non-zero in the sector swept out by the geodesics γ, the mapping $\widetilde{f}$ remains away from infinity there. Thus

$$|d\widetilde{f}| = e^\sigma \leq 2e^\sigma,$$

along γ, and

$$\int_\gamma |d\widetilde{f}|\,|dz| \leq 2l_g(\gamma) \leq \frac{\delta}{2}. \tag{5.1}$$

This implies that

$$\lim_{t\to T_\theta} \widetilde{f}(\gamma(t))$$

exists. We denote the limit by $\widetilde{f}(\gamma(T_\theta))$; it lies on $\partial\Sigma$.

We prove next that $\widetilde{f}(\gamma(T_\theta)) \in \partial\Sigma$ depends continuously on the initial direction θ of the geodesic. Let γ_1, $0 \leq t \leq T_{\theta_1}$ and γ_2, $0 \leq t \leq T_{\theta_2}$, be two geodesic rays starting at z_0 with $\theta_1, \theta_2 \in V$. We need to estimate the distance between $\widetilde{f}(\gamma_1(T_{\theta_1}))$ and $\widetilde{f}(\gamma_2(T_{\theta_2}))$. Let $0 < \tau < \min\{T_{\theta_1}, T_{\theta_2}\}$. Then

$$\begin{aligned}|\widetilde{f}(\gamma_1(T_{\theta_1})) - \widetilde{f}(\gamma_2(T_{\theta_2}))| \leq |\widetilde{f}(\gamma_1(T_{\theta_1})) - \widetilde{f}(\gamma_1(\tau))| + |\widetilde{f}(\gamma_1(\tau)) - \widetilde{f}(\gamma_2(\tau))| \\ + |\widetilde{f}(\gamma_2(T_{\theta_2})) - \widetilde{f}(\gamma_2(\tau))|.\end{aligned}$$

The terms $|\widetilde{f}(\gamma_i(T_{\theta_i})) - \widetilde{f}(\gamma_i(\tau))|$ are dominated by the tails of the integrals in (5.1) which are uniformly bounded by $\delta/2$. Now using the continuity of the length function in the hypothesis (ULP*), there is a τ_0 so that both these terms are small for $\tau_0 \leq \tau < \min\{T_{\theta_1}, T_{\theta_2}\}$ if $|\theta_1 - \theta_2|$ is small. The remaining term can be controlled using the continuity of $\widetilde{f}$ and the fact that $|\gamma_1(\tau) - \gamma_2(\tau)|$ is small if $|\theta_1 - \theta_2|$ is small. These estimates prove that the endpoints $f(\gamma(T_\theta)) \in \partial\Sigma$, γ varying, depend continuously on the initial directions $\theta = \gamma'(0)$.

It remains to show that any point in $\partial\Sigma$ is the image $\widetilde{f}(\gamma(T_\theta))$ as in the construction above. Let $\omega \in \partial\Sigma$ and let $\{w_n\}$ be a sequence of points in Σ which converges to ω. Choose a subsequence, labeled the same way, of $z_n = \widetilde{f}^{-1}(w_n)$ converging to a point $\zeta \in \partial\mathbb{D}$. Let $z_0 \in \mathbb{D}$ be a point of distance $< \delta/8$ from ζ.

Let g_1 be the metric on Σ obtained by pulling back the metric g on $\mathbb{D}$ by $\widetilde{f}^{-1}$. Thus $\widetilde{f}\colon (\mathbb{D}, g) \to (\Sigma, g_1)$ is an isometry. Let let $\Gamma_n(t)$ be the g_1-geodesic joining $\widetilde{f}(z_0) = w_0$ to w_n with $\Gamma_n(0) = w_0$. Another subsequence, again labeled in the same way, of the initial directions $\Gamma_n'(0)$ converges to a direction which determines a geodesic Γ. Let $\gamma = \widetilde{f}^{-1}(\Gamma)$, $\gamma = \gamma(t)$, $\theta = \gamma'(0)$, $0 \leq t \leq T_\theta$. Let $\gamma_n = \widetilde{f}^{-1}(\Gamma_n)$ and let $t_n = l_g(\gamma_n) = l_{g_1}(\Gamma_n)$. Write

$$\begin{aligned}|\widetilde{f}(\gamma(T_\theta)) - w_n| &= |\widetilde{f}(\gamma(T_\theta)) - \widetilde{f}(\gamma_n(t_n))| \\ &\leq |\widetilde{f}(\gamma(T_\theta)) - \widetilde{f}(\gamma(\tau))| + |\widetilde{f}(\gamma(\tau)) - \widetilde{f}(\gamma_n(\tau))| \\ &\quad + |\widetilde{f}(\gamma_n(\tau)) - \widetilde{f}(\gamma_n(t_n))|.\end{aligned}$$

As $\gamma_n'(0) \to \gamma'(0) = \theta$, we conclude for n sufficiently large that $|\widetilde{f}(\gamma(T_\theta)) - w_n|$ can be made arbitrarily small by choosing τ close enough to T_θ. Hence $\omega = \widetilde{f}(\gamma(T_\theta))$. This completes the proof in the case $\delta < \infty$.

We indicate now how the argument should be modified in the complete case $\delta = \infty$. Choose a base point z_0, which is fixed for the entire argument. Let $w_0 = \widetilde{f}(z_0)$. The g_1-geodesic rays from w_0 can be extended indefinitely, and we need to know that they have a limit. Any such ray is the image under $\widetilde{f}$ of a geodesic $\gamma = \gamma(t)$, $\gamma(0) = z_0$. Changing $\widetilde{f}$ by an appropriate Möbius transformation of the range, and maintaining the same notation convention as above, we may assume that $U'(0) \geq c > 0$. Then, as before we have $U(t) \geq b + ct$, $t \geq 0$, and

$$\int_{\gamma} |d\widetilde{f}|\,|dz| < \infty. \tag{5.2}$$

Thus $\lim_{t\to\infty} \widetilde{f}(\gamma(t))$ exists, and we denote if by $\widetilde{f}(\gamma(\infty)) \in \partial\Sigma$.

For the continuity of $\widetilde{f}(\gamma(\infty))$ depending on the initial directions at z_0 we argue as follows. Take a geodesic $\gamma_1(t)$ from z_0. This time we modify $\widetilde{f}$ by a Möbius transformation to change the gradient of $u_{\widetilde{f}}$ at z_0 so that $U'(0) \geq c > 0$ for all rays from z_0 that form an angle of less than $\pi/4$ with $\gamma_1'(0)$. This makes the integrals in (5.2) uniformly bounded over all such rays, and $\widetilde{f}$ uniformly bounded in the sector covered by the rays. From here the proof of continuity, and that all of $\partial\Sigma$ is hit by the $\widetilde{f}(\gamma(\infty))$, is almost identical to the above. Only (ULP) is necessary.

□

6. Extremal Lifts

Definition 6.1. Let f be a harmonic mapping satisfying the hypotheses of Theorem 4.1. We say that $\widetilde{f}$ is an *extremal lift* for (4.1) if the extension of $\widetilde{f}$ to $\overline{\mathbb{D}}$ is not injective on $\partial\mathbb{D}$. A geodesic γ in $\mathbb{D}$ is called an *extremal geodesic* if it joins two points on $\partial\mathbb{D}$ where an extremal lift $\widetilde{f}$ fails to be injective. The lift $\widetilde{f}$ is called non-extremal if it remains injective on $\overline{\mathbb{D}}$.

Definition 6.2. The metric g on $\mathbb{D}$ has the *Boundary Points Joined* property (BPJ) if any two points on $\partial\mathbb{D}$ can be joined by a geodesic which lies in $\mathbb{D}$ except for its endpoints.

We now state

Theorem 6.3. *Let* $\mathbf{g}$ *have the properties (ULP) (or (ULP*)) and (BPJ). Then the following hold.*

*(i) Equality holds in (*4.1*) for an extremal lift along an extremal geodesic.*

(ii) The image $\widetilde{f}(\gamma)$ of an extremal geodesic under the extremal lift $\widetilde{f}$ is a Euclidean circle that is also a line curvature.

(iii) The minimal surface $\Sigma = \widetilde{f}(\mathbb{D})$ is part of a catenoid.

Proof. Let $\widetilde{f}$ be an extremal lift along an extremal geodesic γ in $\mathbb{D}$.

(i) Let $\psi(s) = \widetilde{f}(\gamma(s))$ be the restriction of $\widetilde{f}$ to γ, as considered in the proof of Theorem 4.1. The parameter s ranges over an interval I of length $l = l_{\mathbf{g}}(\gamma) \leq \delta$. It was shown in the proof of the theorem that

$$\mathcal{S}_1\psi \leq \frac{2\pi^2}{\delta^2}\,.$$

We claim that $l_{\mathbf{g}}(\gamma) = \delta$ and that

$$\mathcal{S}_1\psi \equiv \frac{2\pi^2}{\delta^2}\,,$$

$$|I\!I(V,V)|^2 \equiv |K|\ .$$

To prove the claim, suppose that $l < \delta$. Then

$$\mathcal{S}_1\psi \leq \frac{2\pi^2}{\delta^2} < \frac{2\pi^2}{l^2}\,,$$

which would imply by Remark 3.1 that ψ cannot fail to be injective on $\overline{I}$, a contradiction. Hence $l = \delta$. Theorem B now shows that $\mathcal{S}_1\psi \equiv \frac{2\pi^2}{\delta^2}$. The proof of Theorem 4.1 shows now that this is only possible if $|I\!I(V,V)|^2 \equiv |K|$, and hence equality must hold in (4.1) along γ.

(ii) The equation $|I\!I(V,V)|^2 \equiv |K|$, implies that $\widetilde{f}(\gamma)$ must be a line of curvature, which is also a circle by part (i).

(iii) According to the Björling problem (see, *e.g.*, [13], p. 121), there exists a unique minimal surface with a given real-analytic tangent plane along a given real-analytic arc. Because the normal vector to any planar line of curvature of a surface forms a constant angle with the normal to the surface (see, *e.g.*, [14], p. 152), there exists an appropriate circle of revolution of a catenoid that is also a line of curvature, and which forms this same angle. By the above uniqueness result, we conclude that $\widetilde{f}(\gamma)$ must lie on a catenoid.

□

We finish the paper with the following criterion.

Theorem 6.4. *Let $\Sigma \subset \mathbb{R}^3$ be a geodesically convex immersed minimal disk. Suppose that*

$$|K| \leq \frac{4\pi^2}{\delta^2}\,, \tag{6.1}$$

where K stands for the Gaussian curvature and δ for the diameter of Σ. Then the following hold.

(i) The minimal disk Σ is embedded.

(ii) The boundary $\partial\Sigma$ admits a continuous parametrization by the circle $\mathbb{S}^1$.

(iii) Suppose that any two points on $\partial\Sigma$ can be joined by a geodesic Γ contained in Σ, except for its endpoints on $\partial\Sigma$. If $\partial\Sigma$ is not a simple curve, then Σ lies on a catenoid. Equality holds in (6.1) along a geodesic Γ that coincides with the unique geodesic on a catenoid that is a circle of revolution.

(iv) The condition (6.1) is sharp.

Proof. In the proof, we may assume that $\delta < \infty$, for otherwise Σ reduces to a plane.

(i) Let $\widetilde{f}$ be a conformal parametrization of Σ defined on $\mathbb{D}$. We need to show that $\widetilde{f}$ is injective. Because Σ is geodesically convex with diameter δ, it follows that pairs of points in $\mathbb{D}$ can be joined by a geodesic in the metric $e^{2\sigma}\mathbf{g}_0$ of length less than δ. A direct calculation shows that (6.1) corresponds to (4.1), and thus $\widetilde{f}$ is injective.

(ii) As a curve in space, the curvature k of a geodesic $\Gamma \subset \Sigma$ is determined by the second fundamental form II, which is bounded by $\sqrt{|K|}$. Hence all such geodesics have uniformly bounded curvature. Fix a base point $w_0 \in \Sigma$, and consider geodesics with initial point w_0 as a function of the angle $\theta \in \mathbb{S}^1$ on the tangent space at w_0 of the initial direction. Geodesics cannot intersect and will leave any compact subset. Because the curvature k is bounded, it is easy to see that each such geodesic Γ extended maximally will converge to a unique limit point $\zeta_\theta \in \partial\Sigma$. Because two geodesics with sufficiently close initial data remain arbitrarily close on a given compact set, and again because the remaining tails are controlled by the bound on its curvatures, it follows that the limit point ζ_θ depends continuously on θ. It is also easy to see that any point on $\partial\Sigma$ is of this form, proving part (ii) of the theorem.

(iii) Suppose that $\partial\Sigma$ is not simple. Hence there exist two values θ_1, θ_2 for which $\zeta_{\theta_1} = \zeta_{\theta_2}$. We may assume that the correspondence $\theta \to \zeta_\theta$ is one-to-one for $\theta_1 < \theta < \theta_2$. By assumption, for small $\epsilon > 0$, there exists a geodesic Γ_ϵ joining the points $\zeta_{\theta_1+\epsilon}$ and $\zeta_{\theta_2-\epsilon}$, and as $\epsilon \to 0$, Γ_ϵ will converge to a geodesic Γ in Σ closing up at $\zeta_{\theta_1} = \zeta_{\theta_2}$. Then $l(\Gamma) \leq \delta$ and its curvature k satisfies

$$|k| \leq \sqrt{|K|} \leq \frac{2\pi}{\delta}\,.$$

Consider an arc-length parametrization $\varphi : I \to \Gamma$, defined on an interval of length at most δ. Then

$$\mathcal{S}_1\varphi = \frac{1}{2}k^2 \leq \frac{2\pi^2}{\delta^2}\,.$$

As argued in the proof of part (i) of Theorem 6.2, we conclude that $l(\Gamma) = \delta$, $\mathcal{S}_1\varphi \equiv 2\pi^2/\delta^2$, and that Γ is a circle. It is also a line of curvature because II must be maximal long Γ. This proves part (iii).

(iv) The criterion is sharp in the following two senses. First of all, a configuration is possible for which (6.1) holds with equality along a geodesic: consider Σ to be a geodesic ball of radius π centered at a point w_0 on the geodesic of revolution Γ of the catenoid obtained by rotation of the curve $x = \cosh(z)$. The Gaussian curvature along Γ is 1 in absolute value, so equality will hold in (6.1) because $\delta = 2\pi$. Everywhere else on Σ, $|K| < 1$, so the criterion is satisfied.

On the other hand, the criterion is also sharp in the sense that the constant $4\pi^2/\delta^2$ cannot be improved; simply take a geodesic ball as above of radius $r > \pi$ to violate an embedding.

$\square$

7. Extensions to Space

The purpose of this section is to derive an extension of certain lifts to the entire 3-space that represents an analogue of the Ahlfors-Weill construction. The extension will be a consequence of setting up appropriate circle bundles in domain and range that can be matched, for example, by Möbius transformations. By appealing to generalized best Möbius approximations to the lift, a rather explicit extension was obtained in [12] when $\mathbf{g}$ is the Poincaré metric, which under natural additional assumptions, was shown to be quasiconformal. We will not pursue here similar considerations of quasiconformality. To establish the results, we will assume that the conformal metric in $\mathbb{D}$ is *complete* and that it satisfies the conditions (ULP) and (BPJ). We will introduce a variant of the notion of being non-extremal that will be satisfied, for example, whenever strict inequality holds in (4.1), and which will be equivalent to being non-extremal when the metric is real analytic.

Throughout this section, $\widetilde{f}$ will be assumed to satisfy (4.1). Lemma 5.1 ensures that $u_{\widetilde{f}}$ is convex in the metric $\mathbf{g}$, and because this property is based on estimating Ahlfors' Schwarzian, the functions $u_{M\circ\widetilde{f}}$ will also be convex whenever M is a Möbius transformation. We will say that the *unique critical point property* (UCP) holds if for every such shift, the function $u_{M\circ\widetilde{f}}$ exhibits at most one critical point in $\mathbb{D}$. If $\widetilde{f}$ satisfies (4.1) with a strict inequality everywhere, then $u_{\widetilde{f}}$ and all $u_{M\circ\widetilde{f}}$ will be strictly convex. Hence at most one critical point can occur and the (UCP) condition will hold.

We first establish the connections between the (UCP) property and that of being non-extremal. Recall that the lift $\widetilde{f}$ admits a spherically continuous extension to the closed disk.

Lemma 7.1. *Let $\zeta \in \partial\mathbb{D}$ and suppose that $\widetilde{f}(\zeta)$ is a finite point. If γ is a geodesic ray in $\mathbb{D}$ ending at ζ then $\widetilde{f}(\gamma)$ has finite length.*

Proof. Let z_0 be the initial point of γ and let M be a Möbius transformation such that $u_{M\circ\widetilde{f}}$ has positive derivative at z_0 in the direction of γ. It follows as in (5.2) that $M \circ \widetilde{f}(\gamma)$ has finite length. If M fixes infinity, that is, is affine, then$\widetilde{f}(\gamma)$ will also have finite length. On the other hand, if M is an inversion with some center $q \in \mathbb{R}^3$, then $q \neq \widetilde{f}(\zeta)$, for otherwise $M \circ \widetilde{f}(\gamma)$ would have infinite length. We conclude that $\widetilde{f}(\gamma)$ also has finite length.

$\square$

Lemma 7.2. *Suppose that* $\mathbf{g}$ *is real-analytic and that* $\widetilde{f}$ *is non-extremal. Then (UCP) holds.*

Proof. Suppose, by way of contradiction, that (UCP) does not hold. Then there exists M such that $u_{M\circ\widetilde{f}}$ has two critical points, say $z_1, z_2 \in \mathbb{D}$. Because of convexity, $u_{M\circ\widetilde{f}}$ attains its absolute minimum at z_1, z_2 and also along the geodesic segment joining them. Since the quantities involved are real analytic, we conclude that $u_{M\circ\widetilde{f}}$ is constant along the entire geodesic γ through z_1, z_2 extended up to the boundary in both directions to points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$. This implies that, up to a constant factor, $|d(M\circ\widetilde{f})| = e^{\rho}$ along γ, and hence $M\circ\widetilde{f}(\gamma)$ has infinite length in both directions. From Lemma 7.1 we see that $M\circ\widetilde{f}(\zeta_1) = M\circ\widetilde{f}(\zeta_2)$ must be the point at infinity. To conclude that $\widetilde{f}$ is extremal, we must show that the endpoints ζ_1, ζ_2 of the geodesic γ are distinct. Suppose not, and let $D \subset \mathbb{D}$ be the region enclosed by γ and its endpoint $\zeta_1 = \zeta_2$. The all geodesics starting from a fixed point $z_0 \in \gamma$ pointing into D must also converge to ζ_1. If along any such geodesic the function $u_{M\circ\widetilde{f}}$ became eventually increasing, then $M\circ\widetilde{f}$ would be finite at ζ_1, a contradiction. Since $u_{M\circ\widetilde{f}}(z_0)$ is already the minimum, then $u_{M\circ\widetilde{f}}$ must be constant along all such geodesics, or equivalently, $u_{M\circ\widetilde{f}}$ is constant in D. The identity principle show that $u_{M\circ\widetilde{f}}$ is constant in $\mathbb{D}$, and therefore $M\circ\widetilde{f}$ is constant and equal to infinity on $\partial\mathbb{D}$. The original lift $\widetilde{f}$ is also constant on $\partial\mathbb{D}$. If this constant is a finite point then the topological sphere $\widetilde{f}(\overline{\mathbb{D}})$ would exhibit points of positive Gaussian curvature, which is impossible. If the constant is the point at infinity, then the shift M was not necessary to begin with, and $\widetilde{f}$ satisfies (4.1) with $\rho = \sigma$. The inequality forces the Gaussian curvature to be identically zero and we are back to the holomorphic case treated in [7].

□

We now show the complementary implication.

Lemma 7.3. *Suppose that the (UCP) property holds. Then* $\widetilde{f}$ *is not extremal.*

Proof. Suppose that $\widetilde{f}$ is extremal, and let γ be a geodesic in $\mathbb{D}$ joining points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ for which $\widetilde{f}(\zeta_1) = \widetilde{f}(\zeta_2)$. Consider a Möbius shift M sending the common point to infinity. As we have seen before, $u_{M\circ\widetilde{f}}$ must be constant along γ. To prove that the constant value, say c, is the minimum, it suffices to show that $u_{\widetilde{f}}$ has a critical point on γ. Let D_1, D_2 be the two region into which γ divides $\mathbb{D}$, and suppose that $u_{M\circ\widetilde{f}}$ has no critical point along γ. Then the normal derivative of $u_{M\circ\widetilde{f}}$ along γ must keep a constant sign, thus $u_{M\circ\widetilde{f}}$ must be decreasing when moving away from γ in the direction of, say, D_1. In other words, $u_{M\circ\widetilde{f}}(z) \le c$ for all $z \in D_1$, hence $|d\widetilde{f}(z)| \ge (1/c^2)e^{\rho}$ there. This implies that $M\circ\widetilde{f}(\zeta) = \infty$ for all points on the arc $C = \partial\mathbb{D}\cap\partial D_1$. Hence $\widetilde{f}$ itself is constant on C. Since γ is an extremal geodesic, we conclude that $\widetilde{f}(D_1)$ is a minimal surface with boundary a circle. This is readily seen to imply that the minimal surface must reduce to a plane, and we are back in the holomorphic case found in [7] where it is shown that, indeed, c is the minimum value. This finishes the proof of the lemma.

$\square$

Let us now consider the following type of bundles of circles that fibre 3-space. As a general configuration, let B be a smooth, open surface in $\mathbb{R}$, and consider a family $\mathfrak{C}(B)$ of Euclidean circles C_p indexed by $p \in B$, at most one of which is a Euclidean line, having the properties:

(i) C_p is orthogonal to B at p and $C_p \cap \overline{B} = \{p\}$;

(ii) if $p_1 \neq p_2$ then ${}_{p_1} \cap C_{p_2} = \emptyset$;

(iii) $\bigcup_{p\in B} C_p = \mathbb{R}^3 \setminus \partial B$.

We regard the point at ∞ as lying on the line in $\mathfrak{C}(B)$. We refer to $p \in C_p$ as the base point. If B is unbounded then there is no line in $\mathfrak{C}(B)$, for a line would meet $\overline{B}$ at its base point and at the point at infinity, contrary to (i).

The model case is $B = \mathbb{D}$, with $\mathfrak{C}_{\mathrm{o}} = \mathfrak{C}(\mathbb{D})$ being the collection of circles C_z orthogonal to the complex plane passing through $z \in \mathbb{D}$ and its reflection $1/\bar{z}$. In this case, only the circle through the origin becomes a line. In order to set up a bundle of this type when $B = \widetilde{f}(\mathbb{D})$ will require (UCP) to hold. The bundle will be established through a series of lemmas, and it will be necessary to shift the lift $\widetilde{f}$ by suitable Möbius transformations $M = M_q$ of the form

$$M(p) = \frac{p-q}{|p-q|^2}\,,$$

for which the canonical function is given by

$$u_{M\circ\widetilde{f}} = |\widetilde{f} - q|\, u_{\widetilde{f}}\,.$$

Lemma 7.4. *Let $\widetilde{f}$ satisfy (4.1) and let $z_0 \in \mathbb{D}$ be fixed. Consider the set C of points $q \in \mathbb{R}^3$ for which $u_{M\circ\widetilde{f}}$ has a critical point at z_0. Then*

(i) C is a circle orthogonal to Σ at $\widetilde{f}(z_0)$ with radius $r(z_0) = \dfrac{e^{\sigma(z_0)}}{|\nabla \log u_{\widetilde{f}}(z_0)|}$;

(ii) C is symmetric with respect to the tangent plane to Σ at $\widetilde{f}(z_0)$;

(iii) $(C \setminus \{\widetilde{f}(z_0)\}) \cap \overline{\Sigma} = \emptyset$.

Proof. We show first the following basic fact. Let D be a planar domain and let e^{τ} be a given positive function on D. Then the set of points $q \in \mathbb{R}^3$ for which $e^{\tau}|q-z|^2$ has a critical point at $z_0 \in D$ is a circle orthogonal to the plane passing through the points z_0 and $z_0 + 1/\tau_z(z_0)$. Indeed, the critical point condition can be written in the form

$$\tau_{\bar{z}}(z_0) = \frac{p - z_0}{|q - z_0|^2}\,,$$

where p is the projection of q onto the plane on which D lies. This last equation is readily seen to be equivalent to the condition

$$\left| q - z_0 - \frac{1}{2\tau_z(z_0)} \right|^2 = \left| \frac{1}{2\tau_z(z_0)} \right|^2 ,$$

which represents the claimed circle.

Since the statement in (i) involves only first order data, we may replace the minimal surface $\widetilde{f}(\mathbb{D})$ with the projection onto the tangent plane at $\widetilde{f}(z_0)$ of a neighborhood of $\widetilde{f}(z_0)$ on the surface. This projection is planar domain D as above, and the result follows after considering $u_{M\circ\widetilde{f}}$ as a function of the image point $w = \widetilde{f}(z)$. This proves parts (i) and (ii).

To prove (iii) let $q \in C$, $q \neq \widetilde{f}(z_0)$. Then $u_{M\circ\widetilde{f}}$ is convex and has a positive minimum point at z_0. If q were on Σ or on its boundary, then $u_{M\circ\widetilde{f}}$ would tend there to zero, a contradiction.

□

For $z \in \mathbb{D}$ and $w = \widetilde{f}(z)$, the circle described in the lemma will be denoted by C_w. Through the following lemmas it will be shown that the family of circles $\{C_w\}_{w\in\Sigma}$ constitutes a circle bundle of $\mathbb{R}^3$ with base Σ. This bundle will be denoted simply by $\mathfrak{C}$. An important property is that, in the presence of a critical point in $\mathbb{D}$, the radius $r(z)$ of the circle $C_{\widetilde{f}(z)}$ tends to zero as z approaches $\partial\mathbb{D}$. Indeed, we claim that for all points z away from the unique critical point of $u_{\widetilde{f}}$ we will have

$$e^{-\rho}|\nabla u_{\widetilde{f}}| \geq a > 0\,,$$

for some absolute constant a. This estimate is direct consequence of the lower bound by linear growth $u_{\widetilde{f}}$ along geodesics rays emanating from the critical point. From this it follows that

$$r(z) \leq \frac{1}{a u_{\widetilde{f}}}\,, \tag{7.1}$$

which tends to zero at the boundary.

Lemma 7.5. *Let $\widetilde{f}$ satisfy (4.1). If the condition (UCP) holds then Σ is bounded if and only if $u_{\widetilde{f}}$ has a critical point in $\mathbb{D}$.*

Proof. Suppose first that $u_{\widetilde{f}}$ has a critical point, say $z_0 \in \mathbb{D}$. The (UCP) property implies that $u_{\widetilde{f}}$ must be strictly increasing along any geodesic ray γ starting at z_0. Hence, as in (5.2), we conclude that $\widetilde{f}$ remains bounded on γ, with bounds depending on the constants b, c. Since these constant can be chosen independently on the direction of γ at z_0, we see that $\widetilde{f}$ is bounded.

Suppose now that Σ is bounded. Then Lemma 7.1 shows that $\widetilde{f}(\gamma)$ has finite length for any geodesic ray γ reaching $\partial\mathbb{D}$. This implies that along any such ray starting, say, at the origin, $u_{\widetilde{f}}$ must become eventually increasing. Therefore, $u_{\widetilde{f}}$ attains an interior minimum, proving the existence of the desired critical point.

$\square$

The lemma can equally well be stated for any Möbius shift $M \circ \widetilde{f}$, in particular, if $M(\Sigma)$ is bounded then $u_{M\circ\widetilde{f}}$ must have a critical point in $\mathbb{D}$. We can now show that $\mathfrak{C}$ is a circle bundle of $\mathbb{R}^3$ with base Σ. Since the property (i) of such a bundle is met by Lemma 7.4, we are to show the properties (ii) and (iii) of a circle bundle.

For (ii), let C_{w_1}, C_{w_2} be two such circles having a common point q. If $q \in \Sigma$ then $q = w_1 = w_2$, hence the circles are the same. If $q \notin \Sigma$ then $u_{M_q\circ\widetilde{f}}$ has a critical point at $z_1 = \widetilde{f}^{-1}(w_1)$ and at $z_2 = \widetilde{f}^{-1}(w_2)$. The (UCP) property implies that $z_1 = z_2$, hence $C_{w_1} = C_{w_2}$.

For (iii), consider a point $q \notin \overline{\Sigma}$. Then $M_q(q) = \infty \notin M_q(\overline{\Sigma})$, meaning that $M_q(\Sigma)$ must be bounded. Lemma 7.5 implies that $u_{M_q\circ\widetilde{f}}$ has a critical point in $\mathbb{D}$ and therefore $q \in C_w$ for some $w = \widetilde{f}(z)$.

We make two final observations before setting up the extension. First, it follows from part (i) of Lemma 7.4 that $C_w \in \mathfrak{C}$ becomes a line exactly when $\widetilde{f}^{-1}(w)$ is the unique critical point of $u_{\widetilde{f}}$. Secondly, that the bundle $\mathfrak{C}$ with base $\Sigma = \widetilde{f}(\mathbb{D})$ is *conformally natural*, in the following sense. For a fixed Möbius mapping M_0, Lemma 7.4 produces a bundle $\mathfrak{D}$ over the base $M_0(\Sigma)$, which is not difficult to see coincides with the collection of circles $M_0(C)$ for $C \in \mathfrak{C}$.

Let now $\widetilde{f}$ be a lift satisfying (4.1) for which (UCP) holds. It is natural to consider a spatial extension of the lift by matching the circles in the bundles $\mathfrak{C}_0$ and $\mathfrak{C}$ through the respective base points. For the actual pointwise correspondence between C_z and $C_{\widetilde{f}(z)}$ we use an adequate affine mapping D_z with $D_z(z) = \widetilde{f}(z)$. With this, let $\mathcal{E}_{\widetilde{f}} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\mathcal{E}_{\widetilde{f}}(p) = \begin{cases} \widetilde{f}(z) & , \quad p = z \in \overline{\mathbb{D}} \\ \\ D_z(p) & , \quad p \in C_{\widetilde{f}(z)} \end{cases} . \tag{7.2}$$

The mapping $\mathcal{E}_{\widetilde{f}}$ is injective on $\overline{\mathbb{D}}$ because $\widetilde{f}$ is non-extremal. The properties of circle bundles guarantees that $\mathcal{E}_{\widetilde{f}}$ is injective elsewhere and also onto. It is also readily seen to be continuous at all points $p \notin \partial\mathbb{D}$, whereas the continutity on $\partial\mathbb{D}$ is guaranteed by (7.1) when Σ is bounded. Since the bundle $\mathfrak{C}$ transforms to the corresponding bundle over the base $M \circ \widetilde{f}$ for any Möbius shift, we conclude that the extension $\mathcal{E}_{\widetilde{f}}$ is continuous in the spherical metric whether Σ is bounded or not.

As a final comment, we mention that $\mathcal{E}_{\widetilde{f}}$, when restricted to points $p \in \mathbb{C}$ does give back the Ahlfors-Weill extension when $\mathbf{g}$ is the Poincaré metric [12].

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